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FAST TRACK COMMUNICATION

String-nets, single- and double-stranded quantum loop gases for non-Abelian anyons

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Abstract
String-nets and quantum loop gases are two prominent microscopic lattice models to describe topological phases. String-net condensation can give rise to both Abelian and non-Abelian anyons, whereas loop condensation usually produces Abelian anyons. It has been proposed, however, that generalized quantum loop gases with non-orthogonal inner products could support non-Abelian anyons. We detail an exact mapping between the string-net and these generalized loop models and explain how the non-orthogonal products arise. We also introduce an equivalent loop model of double-stranded nets where quantum loops with an orthogonal inner product and local interactions supports non-Abelian Fibonacci anyons. Finally, we emphasize the origin of the sign problem in systems with non-Abelian excitations and its consequences on the complexity of their ground state wavefunctions.

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Some strongly correlated quantum many body systems display a type of order which is topological in nature and cannot be characterized by local order parameters [1]. The Landau paradigm of phase transitions is based on the existence of long-range order for some local order parameter (e.g. a local magnetization); different phases can be characterized by their symmetry properties according to the configuration of such order parameter (e.g. a symmetric phase with vanishing magnetization and a symmetry-breaking phase with magnetization along a well-defined direction). Topological phases, instead, lack long-range order and cannot be classified depending on the symmetry properties of a local order parameter. Instead, different topological phases can be identified by the degeneracy of their ground states in dependence on the global topology of the space where the system lives or by the different fractional statistics of their excitations.

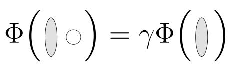
Much of the recent interest in topological phases is due to the possibility of braiding these topological excitations to perform quantum computations and to exploit their confinement to store quantum information encoded into topological degrees of freedom, whose non-locality makes them intrinsically robust against local perturbations [7, 8]. Fractional quantum Hall systems are the primary experimental examples of topological order, so far. They display a robust ground state degeneracy which cannot be lifted by any local perturbation [2, 3] and fractionalized degrees of freedom [4–6].

Both to understand naturally occurring phases and to engineer new systems displaying topological behavior, microscopic models have been proposed. Topological phases with Abelian excitations in two spatial dimensions can be constructed from loop models whose ground states are condensates of fluctuating non-intersecting closed loops. Non-Abelian phases, and among these the ones providing a physical support for *universal* quantum computation, are more difficult to attain. One proposal for a microscopic model is to consider string-nets where the dynamical degrees of freedom are strings which allow for branchings and intersections [9]; alternatively, Fendley has recently suggested that non-Abelian phases could also be obtained from quantum gases of non-intersecting loops, if one allows for non-orthogonal inner products between classical loop configurations [10, 11].

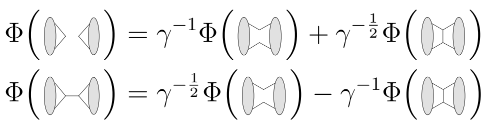
In this communication we demonstrate explicitly that such a quantum loop gas formulation follows directly from the string-net fusion rules, establishing precisely the connection between the two approaches. We also introduce an example of a *loop* model with an *orthogonal* inner product supporting non-Abelian excitations. Finally we show that attempts to endow the loops with a dynamics in the manner of Rokhsar–Kivelson, using simple moves between pairs of loop configuration, are doomed to fail. In fact, the Hamiltonians are plagued by a 'sign problem' which hinders a simple study of non-Abelian topological states using, for instance, standard Monte Carlo methods.

1. String-net rules as projectors

Systems whose excitations are Fibonacci anyons provide the simplest setting capable of supporting *universal* quantum computations. In the string-net picture, the ground state wavefunction for Fibonacci anyons [12–14] is a linear combination of classical string-net configurations; the amplitude of each classical configuration in the quantum ground state is determined in such a way that its wavefunction Φ satisfies the following constraints [9]:



(1)



(2)

where $\gamma = \frac{1+\sqrt{5}}{2}$ is the golden ratio and the relation $\gamma^2 = \gamma + 1$ holds. The gray ovals represent undetermined but fixed classical string-net configurations. The constraint in (1), for instance, ensures that the weight in the ground state wavefunction of any given classical configuration is $1/\gamma$ times the weight of the same classical configuration with the addition of a single loop. Imposing (1) on the ground state wavefunction enforces then a loop fugacity. Analogously, the two constraints in (2) enforce the fusion rules involving four anyons. The 'surgery' relations can also be inverted to yield

$$\begin{aligned}\Phi(\,) &= \gamma^{-1}\Phi(\,) + \gamma^{-\frac{1}{2}}\Phi(\,)\\ \Phi(\,) &= \gamma^{-\frac{1}{2}}\Phi(\,) - \gamma^{-1}\Phi(\,)\end{aligned} \tag{3}$$

Looking for an operator implementing the relations above in a lattice system, we now consider (2) and (3) together, rescale them by a factor $-1/2$, and re-express them in matrix form as

$$F\,\mathbf{v} = \frac{1}{2}\begin{pmatrix} 1 & -\gamma^{-1} & 0 & -\gamma^{-\frac{1}{2}} \\ -\gamma^{-1} & 1 & -\gamma^{-\frac{1}{2}} & 0 \\ 0 & -\gamma^{-\frac{1}{2}} & 1 & \gamma^{-1} \\ -\gamma^{-\frac{1}{2}} & 0 & \gamma^{-1} & 1 \end{pmatrix}\mathbf{v} = 0, \tag{4}$$

where $\mathbf{v} = v_1|\rangle\langle\rangle + v_2|\asymp\rangle + v_3|\rangle\!\!-\!\!\langle\rangle + v_4|\asymp\!\!\!\!|\,\rangle$; v_1, v_2, v_3, v_4 are the amplitudes $\Phi(\,), \Phi(\,), \Phi(\,), \Phi(\,)$ respectively and the vectors $\{|\rangle\langle\rangle, |\asymp\rangle, |\rangle\!\!-\!\!\langle\rangle, |\asymp\!\!\!\!|\,\rangle\}$, labeled by classical string-net configurations, are assumed to be an *orthonormal* basis of the Hilbert space. F is Hermitian and $F^2 = F$, hence F is a projector; the eigenvalues are $\{0, 0, 1, 1\}$ with corresponding eigenvectors $\{|g_1\rangle, |g_2\rangle, |e_3\rangle, |e_4\rangle\}$. By construction, the two-dimensional ground state spanned by $|g_1\rangle$ and $|g_2\rangle$ is a quantum superposition of classical string-net states satisfying the rules in (2) or (3), whereas the excited states $|e_3\rangle$ and $|e_4\rangle$ violate such rules. The local four-dimensional projector in (4) is the minimal operator annihilating both $|g_1\rangle$ and $|g_2\rangle$ and the ground state of the Fibonacci string-net model must be annihilated by a sum of these projectors implementing all the possible local surgeries.

2. Quantum loop models and non-orthogonal basis vectors

The four-dimensional formalism described above can be reduced in a natural way to a two-dimensional one describing the appropriate Hilbert space for quantum loop models [10]. Exploiting the identity $\frac{\gamma}{\sqrt{\gamma+1}} = 1$, we can multiply the off-diagonal elements of the matrix F in (4) by $\frac{\gamma}{\sqrt{\gamma+1}}$ and rewrite it as

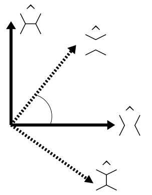
$$F_{\alpha=\gamma} = \frac{1}{2}\begin{pmatrix} 1 & -\sqrt{\frac{1}{1+\gamma}} & 0 & -\sqrt{\frac{\gamma}{1+\gamma}} \\ -\sqrt{\frac{1}{1+\gamma}} & 1 & -\sqrt{\frac{\gamma}{1+\gamma}} & 0 \\ 0 & -\sqrt{\frac{\gamma}{1+\gamma}} & 1 & \sqrt{\frac{1}{1+\gamma}} \\ -\sqrt{\frac{\gamma}{1+\gamma}} & 0 & \sqrt{\frac{1}{1+\gamma}} & 1 \end{pmatrix}. \tag{5}$$

Such a form is convenient for replacing the golden ratio γ by an arbitrary *positive* real number α (a choice of inner product for loop states will prove equivalent to fixing α). F_α is still a projector with eigenvalues $\{0, 0, 1, 1\}$ and corresponding (α-dependent) orthonormal eigenvectors $\{|g_1\rangle, |g_2\rangle, |e_3\rangle, |e_4\rangle\}$:

$$\begin{pmatrix} g_1 \\ g_2 \\ e_3 \\ e_4 \end{pmatrix} = \begin{pmatrix} a & a & b & b \\ -c & c & d & -d \\ -c & -c & d & d \\ a & -a & b & -b \end{pmatrix}\begin{pmatrix} \rangle\langle \\ \asymp \\ \rangle\!\!-\!\!\langle \\ \asymp\!\!\!\!| \end{pmatrix}, \tag{6}$$

$$a = \frac{\sqrt{1+\alpha}+1}{2\sqrt{1+\alpha+\sqrt{1+\alpha}}}; \qquad b = \frac{\sqrt{\alpha}}{2\sqrt{1+\alpha+\sqrt{1+\alpha}}};$$
$$c = \frac{\sqrt{1+\alpha}-1}{2\sqrt{1+\alpha-\sqrt{1+\alpha}}}; \qquad d = \frac{\sqrt{\alpha}}{2\sqrt{1+\alpha-\sqrt{1+\alpha}}}.$$

The ground state of F_α is a two-dimensional linear subspace of the original four-dimensional space. It is then natural to study the fate of the four orthonormal basis vectors $\{|\ \rangle, |\ \rangle, |\ \rangle, |\ \rangle\}$ when projected onto the two-dimensional plane spanned by $|g_1\rangle$ and $|g_2\rangle$. Such projected states will be called 'shadow' states and marked by a hat. Inverting (6) and rescaling by a factor of $\sqrt{2}$ to yield normalized shadow states, we obtain



$$\begin{pmatrix} \hat{\ } \\ \hat{\ } \\ \hat{\ } \\ \hat{\ } \end{pmatrix} = \sqrt{2}\begin{pmatrix} a & -c \\ a & c \\ b & d \\ b & -d \end{pmatrix}\begin{pmatrix} g_1 \\ g_2 \end{pmatrix}. \tag{7}$$

The inner product of the original four-dimensional Hilbert space naturally induces an inner product on the two-dimensional plane which, for every $\alpha \geqslant 0$, acts on the shadow vectors as

$$\langle \hat{\ } | \hat{\ } \rangle = 2(ab - cd) = 0$$
$$\langle \hat{\ } | \hat{\ } \rangle = 2(ab - cd) = 0$$
$$\langle \hat{\ } | \hat{\ } \rangle = 2(a^2 - c^2) = \frac{1}{\sqrt{1+\alpha}}$$
$$\langle \hat{\ } \ \hat{\ } \rangle = 2(b^2 - d^2) = \frac{1}{\sqrt{1+\alpha}}.$$

Only two shadow states are necessary for a basis of the two-dimensional plane. One possibility is to choose one of the two orthogonal pairs $\{|\hat{\ }\rangle, |\hat{\ }\rangle\}$ and $\{|\hat{\ }\rangle, |\hat{\ }\rangle\}$. Alternatively, considering only states without branchings, the choice $\{|\hat{\ }\rangle, |\hat{\ }\rangle\}$ represents the natural basis of a quantum loop model. The substitution $\sqrt{1+\alpha} \leftrightarrow d = \lambda^{-1}$ (d and λ defined in [10]) provides the dictionary for establishing the exact correspondence between the quantum loop states described in [10] and the Fibonacci string-net states of [9]. Hence relations (2) and (3) can be interpreted either as relations between *amplitudes* in the four-dimensional space where $\{|\ \rangle, |\ \rangle, |\ \rangle, |\ \rangle\}$ are orthonormal, or as relations between shadow *basis vectors* in the two-dimensional space where $\{|\hat{\ }\rangle, |\hat{\ }\rangle, |\hat{\ }\rangle, |\hat{\ }\rangle\}$ are not orthogonal.

3. Lattice implementation

Having proven the equivalence of string-nets and quantum loop gases in the continuum, we now discuss their *microscopic* correspondence in a specific lattice system. We focus on a simple fully packed model on the square lattice, where four tile configurations are allowed , each tile representing a possible surgery point. By introducing suitable dynamical rules we will show how this tile model can implement the Levin–Wen string-net model for Fibonacci anyons [9]. We will then introduce two mappings: one into a non-orthogonal loop model in the manner of Fendley and one into a loop model with an *orthogonal* inner product.

The Hilbert space of an N-tile system is the N-fold tensor product of four-dimensional local Hilbert spaces and the string-net rules can be applied at the single plaquette level so that the transition amplitudes between plaquette configurations are given by F_α in (5):

$$H_F = J\,(\mathbb{I} \otimes \cdots \otimes \mathbb{I} \otimes F_\alpha + \cdots + F_\alpha \otimes \mathbb{I} \otimes \cdots \otimes \mathbb{I})\,. \tag{8}$$

By applying (8), the 4^N-dimensional Hilbert space is projected onto the 2^N-dimensional Hilbert space of N decoupled spins 1/2, consistently with the fact that F_α leaves two effective degrees of freedom per tile. The redundancy of keeping four states per plaquette will be useful in the following to construct the mappings between string-net and loop models. Note that enforcing the topological rules at the single-tile level through (8) cannot be sufficient to drive the system into a topological phase since the full-packing condition in the model does not allow a string configuration on a single tile to stretch and spread its topology over more tiles. The role of operators acting on several plaquettes at the same time, such as the the 12-spin operators in the following, is to enforce topological constraints on the degrees of freedom emerging from the fully packed background so that the topological features of a configuration can propagate throughout the system. This turns out to be the key feature in arguing that *local* lattice operators are sufficient to endow the system with topological properties at large scales. Figures 1(a)–(c) show how the Levin–Wen string-net model defined on the honeycomb lattice [9] can be represented in our square tile language. Down-spins (dark-blue circles) represent string segments and up-spins (pink circles) represent empty edges. The string-net configurations are constrained by 3-spin 'star' operators (imposing the fusion rules) and the dynamics is implemented through 12-spin interactions (red dashed lines in figure 1). After deforming the honeycomb lattice to a brick wall lattice, we obtain a square lattice by adding edges populated by dummy up-spins (sky-blue circles) (figure 1(b)). Re-drawing the lattice so that the spins are centered on the tiles, we generate the square lattice on which the string-net tile model is defined (figure 1(c)). A subtlety worth noting is that the 12-spin and 3-spin operators can be translated only by an even number of lattice spacings on the square lattice, so that all the operators commute with each other and the dummy spins never take part in the dynamics.

The mapping to a loop model with non-orthogonal inner product proceeds from figure 1(c) by choosing local reference systems in a checkerboard fashion, alternating $\{|\ \rangle, |\ \rangle\}$ and $\{|\ \rangle, |\ \rangle\}$ as basis vectors. Then, the orthogonal spin-up and spin-down states (pink circles, dark-blue circles) can be mapped into the orthogonal states or respectively, outlining on the tiles the topology of the original string-net with dressed vertices (figure 1(d)). By a simple change of basis to the non-orthogonal pair $\{|\ \rangle, |\ \rangle\}$, as explained in the first part of the communication, any configuration can finally be written as a linear combination of loop configurations with the correct amplitudes. Figure 1(e) would be one configuration in such an expansion for the configuration in figure 1(d).

It has been argued [11] that loop models with orthogonal inner products and local interactions do not support non-Abelian excitations. In figure 1(f) we simply re-write the string-net tile model of figure 1(c) in terms of loops by choosing local reference systems in a checkerboard fashion, alternating and as pairs of basis vectors representing the spin-up and spin-down states. In contrast to what we had so far, here and are the *orthogonal* states and they have *no* relation with the two states $\{|\ \rangle, |\ \rangle\}$ and the projector (8). As before, the 12-spin and 3-spin operators can be translated only by an even number of lattice spacings to ensure the exact correspondence with the original string-net model on the honeycomb lattice. Note how, geometrically, figure 1(f) is nothing but figure 1(a) where the strings forming the net are now ribbons and small loops fill the empty spaces as expected

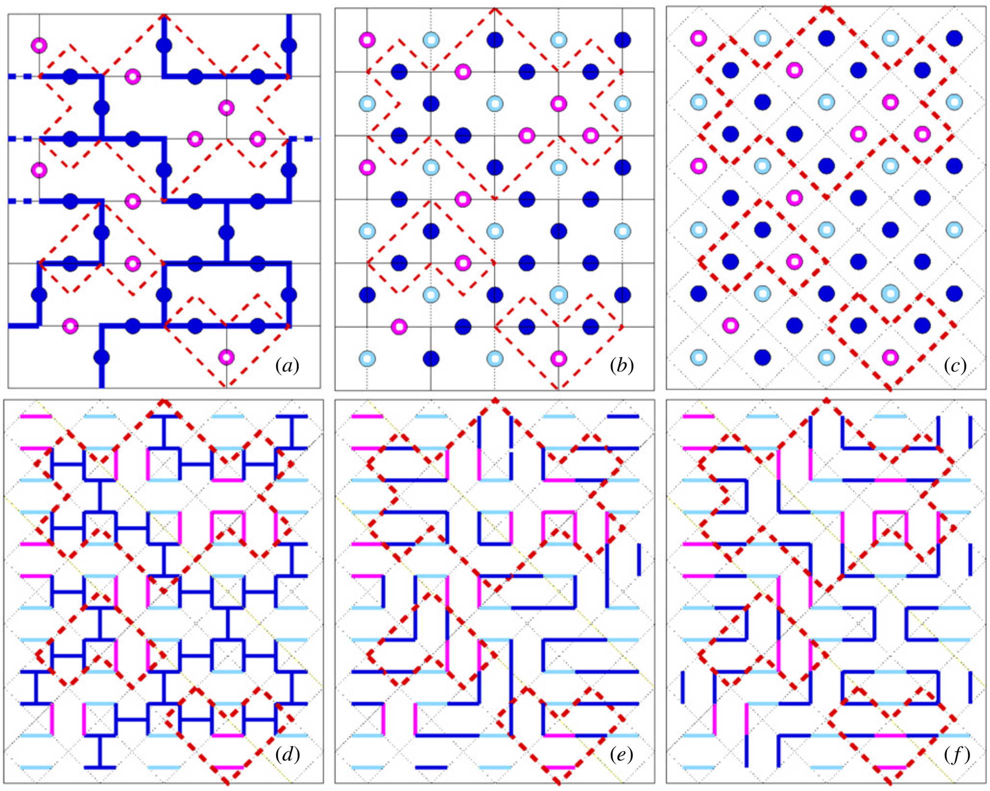


Figure 1. Exact mapping from a string-net model on the honeycomb lattice to a fully packed quantum loop model on the square lattice. Details in the text.

(This figure is in colour only in the electronic version)

from a fully packed model. This last mapping shows explicitly how quantum loop gases with orthogonal inner products *and* local interactions (involving not one, but a finite number of plaquettes) allow for non-Abelian excitations. In a continuum setting string-nets are then substituted by strongly attracting loops so that each string-net segment is replaced by double-stranded loop segments. The dynamics of the loops is by the construction identical to the one of string-nets.

4. The sign problem and quantum interference

Although we can now write models of orthogonal loops for non-Abelian anyons, the complexity of the Fibonacci anyons is not overcome, as emphasized in this last section in terms of RK decompositions and sign problems.

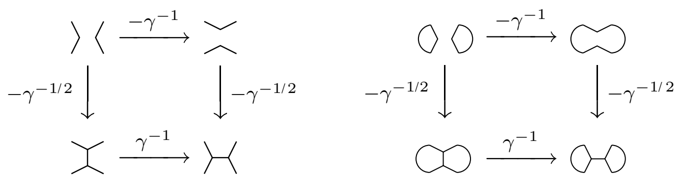
Many Hamiltonians admit a Rokhsar–Kivelson (RK) decomposition [15]. This means that they can be written as a weighted sum of projectors $P_{\{c\}}$ connecting the the configurations in the set $\{c\} = \{c_1, \ldots, c_n\}$ for some finite integer n:

$$H = \sum_{\{c\}} w_{\{c\}} P_{\{c\}}.$$

Being RK-decomposable is a basis-dependent property; however, oftentimes the basis elements are labeled by local degrees of freedom (such as the states of lattice sites, edges or plaquettes) in a natural way; the dynamics can then be implemented in terms of operators connecting the configurations in $\{c\}$ which differ only by one localized degree of freedom. In those cases it is straightforward to write the Hamiltonian in the RK form and a unique ground state can be constructed as the state annihilated by all the projectors $P_{\{c\}}$. If all the projectors have a finite weight $w_{\{c\}}$, then the system is necessarily gapped since any excited state is annihilated by some but not all projectors.

To each Hamiltonian decomposable in terms of 2×2 projectors one can associate a corresponding classical system [16–19] and the fact that the ground state is annihilated by projectors implies detailed balance conditions for the transition rates between the corresponding classical configurations. Kitaev's toric code [7] (with its Abelian anyons) provides such an example; even though it admits a different RK decomposition for each of its four degenerate ground states, the degeneracy is detectable only by non-local operators and the ground states are dynamically disconnected in the thermodynamic limit. For the Fibonacci anyons case, instead, each *local* surgery point has a two-fold degenerate ground state (5). Indeed, two non-compatible RK decompositions are possible for the matrix F_α in (5): one with 2×2 projectors annihilating $|g_1\rangle$ and the other with projectors annihilating $|g_2\rangle$. As a consequence constructing the ground state wavefunction by following a sequence of states related to each other by simple detailed balance relations is a non-trivial task. This is a simple way to view the complexity of the Fibonacci wavefunction as compared e.g. to the toric code, whose Hamiltonian can be written as the sum of two-dimensional projectors and related to a classical stochastic system satisfying detailed balance.

A related feature of the 4×4 Fibonacci projector (5) is that not all of its off-diagonal matrix elements have the same sign. Such a 'sign problem' is the hallmark of quantum–mechanical interference effects as follows:



(9)

The transition amplitudes above are read off directly from F in (4). In the diagram on the right, a choice of boundary conditions has been made: states with and without tadpoles (lower-right corner and upper-left corner respectively) are connected by multiple paths which interfere destructively. The effect of the 'sign problem' is therefore a *dynamical* partitioning of the Hilbert space into different topological sectors. Note that since $F^n = F$, the interference between $\rangle\langle$ and $\rangle\!\!-\!\!\langle$ (and between $\genfrac{}{}{0pt}{}{\smile}{\frown}$ and $\genfrac{}{}{0pt}{}{\frown}{\smile}$) is totally destructive at every order in F and, as a consequence, any argument about the ergodicity of the dynamics cannot rely on purely classical considerations. At the same time, the sign problem spoils the mapping [16] of RK Hamiltonians to classical–statistical–mechanical systems since a correspondence between the negative off-diagonal elements of the quantum Hamiltonian and the (necessarily positive) classical transition rates cannot be enforced.

5. Conclusions

We have detailed a mapping from string-nets to quantum loop gases with non-orthogonal inner products, showing the equivalence of two apparently diverse microscopic models in the literature of topological phases. We discussed how the non-orthogonality of quantum loop states arises in a transparent way when hard topological constraints are present and the description is restricted to a Hilbert space of reduced dimensionality. We also introduced a third, equivalent, description in terms of quantum loop gases with an *orthogonal* inner product. Indeed, non-orthogonal inner products for quantum loops gases are not necessary to generate non-Abelian anyons and orthogonal inner products can be used as well if the appropriate dynamics involving 12 spins is defined. One of the reasons which led researchers to introduce loop models instead of string-nets to describe non-Abelian anyons was the possibility of considering interactions involving only a limited number of spins. These attempts have so far been unsuccessful and our study establishes that, in loop models as well as for string-nets, excitations with non-Abelian statistics do appear in models with 12-spin interactions. Whether or not non-Abelian phases can be described with interactions involving fewer than 12 spins remains an open problem.

Acknowledgments

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